

CONSTRUCTION OF SEMI - FREE S^3 ACTIONS ON HOMOTOPY SPHERE WITH UNTWISTED FIXED POINT SET

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Abstract

William Browder in his paper "Surgery and the theory of differentiable transformation groups" developed surgery techniques to study semi-free actions of S^1 on homotopy spheres, under the additional assumption that the fixed point set is a homotopy sphere. He used this surgery to show how to construct such actions. In this paper, I discussed a similar theory for semi-free actions of S^3 on homotopy spheres. An open problem is raised at the end of the paper.

Keywords: Free and semi free actions, quaterionic projective space, homotopy and homology spheres, principal fibration, tangent and normal bundles, exact homology and cohomology sequences, lefschetz duality, Poincare duality, Hurwicz homomorphism, Hopf fibration, h - cobordism, equivariant diffeomorphism, VanKampens theorem, Mayer - Vietoris sequence, untwisted fixed point set, Künneth formula, surgery techniques.

1 Introduction

Through this paper, R^n denotes the Euclidean n -space, S^n denotes the unit n -sphere in R^{n+1} and $QP(n)$ the quaterionic projective space, all having the usual differentiable structures. By a homotopy n -sphere abbreviated by Σ^n , we mean a closed differentiable n -manifold having the homotopy type of S^n , and by a homotopy quaterionic projective n -space, abbreviated by $HQP(n)$, we mean a closed differentiable $4n$ -manifold having the homotopy type of $QP(n)$. $\pi_n(M)$ denotes the n^{th} -homotopy group of M , $H_i(M, G)$ and $H^i(M, G)$ denote the homology and cohomology of a space M , with coefficients in the group G and assumed to be satisfying the Eilenberg-Steenrod axioms, See [[15], p.6]. If $G = \mathbb{Z}$, we write $H^i(M)$ and $H_i(M)$ for $H^i(M, \mathbb{Z})$ and $H_i(M, \mathbb{Z})$ respectively. It is well known that S^1 and S^3 are the only compact connected Lie groups which have free differentiable actions on homotopy spheres [4, 5]. It follows from Gleason's lemma [1] that such an action is always a principal fibration which is homotopically equivalent to the classical Hopf fibration.

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In fact, there are always infinitely many differentiably distinct free actions of S^3 on Σ^{4n+3} for $n \geq 2$ [17].

2 Construction of semi-free actions of S^3 :

An action (G, M, ϕ) is called semi-free, if it is free outside the fixed point set, that is $\phi : G \times M \rightarrow M$, and F is the fixed point set of the action, ϕ is semi-free action if $\phi(g, x) = x$, for some $x \in M - F$ then $g = e$ the identity of G . Notice that, there are two types of orbits, fixed points, and G .

Lemma 2.1 Let $\phi : S^i \times M \rightarrow M$ ($i = 1, 3$), be a semi free differentiable action. let F^k denote the union of the k -dimensional components of the set of all fixed points of ϕ . Then the normal bundle of an imbedding $F^k \subset M$ has naturally a complex structure for $i = 1$ and a quaterionic structure for $i = 3$ and that the induced S^i -action on the normal bundle is a scalar multiplication. For proof see [3].

It follows from this lemma 2.1, that the codimension of each component of F in M is even for $i = 1$ and is divisible by 4 for $i = 3$. We shall study the situation where (S^3, Σ^m, ϕ) is a semi-free differentiable action on a homotopy sphere Σ^m and the fixed point set is a homotopy sphere Σ^r . Let (S^3, Σ^m, ϕ) be a semi-free action with fixed point set $\Sigma^r \subset \Sigma^m$, then S^3 acts freely outside Σ^r and S^3 acts freely and linearly on the normal space to Σ^r at each point of Σ^r . See [[10],p.58]. By lemma 2.1, the normal bundle of Σ^r has a quaterionic structure and $m - r = 4k$, $k \geq 1$. Let μ be the (quaterionic) bundle over Σ^r defined by the action. We prove the following :

Theorem 2.1 If (S^3, Σ^m, ϕ_1) and (S^3, Σ^m, ϕ_2) are equivalent, then F_1 is diffeomorphic to F_2 and μ_1 is equivalent to μ_2 , where F_1 and F_2 are the fixed point sets of ϕ_1 and ϕ_2 respectively, μ_1 and μ_2 are the normal bundles of F_1 and F_2 respectively.

Proof. Since (S^3, Σ^m, ϕ_1) and (S^3, Σ^m, ϕ_2) are equivalent then there exists an equivariant diffeomorphism f such that the following diagram commutes :

$$\begin{array}{ccc} S^3 \times \Sigma^m & \xrightarrow{\phi_1} & \Sigma^m \\ I \times f \downarrow & & f \downarrow \\ S^3 \times \Sigma^m & \xrightarrow{\phi_2} & \Sigma^m \end{array}$$

Let $x \in F_1$, i.e, $\phi_1(q, x) = x \quad \forall q \in S^3$ then $f \circ \phi_1(q, x) = \phi_2 \circ (I \times f)(q, x) \quad \forall q \in S^3$, thus $f(x) = \phi_2(q, f(x)) \quad \forall q \in S^3$ implies $f(x) \in F_2$ hence $f(F_1) \subseteq F_2$.

Now, assume that $y \in F_2$, i.e., $\phi_2(q, y) = y \quad \forall q \in S^3$, but f is an equivariant diffeomorphism, then $\exists x \in \Sigma^m$ such that $y = f(x)$ and $f \circ \phi_1(q, x) = \phi_2 \circ (I \times f)(q, x) \quad \forall q \in S^3$, then $f(\phi_1(q, x)) = \phi_2(q, f(x)) = f(x)$, but f is 1-1, we get $\phi_1(q, x) = x \quad \forall q \in S^3$, hence $x \in F_1$ and $f(x) \in f(F_1)$, therefore $f(F_1) = F_2$. Moreover, the equivalence $f : \Sigma^m \rightarrow \Sigma^m$ defines a quaterionic map of the normal bundles μ_1 and μ_2 of F_1 and F_2 respectively, so they are equivalent.

Now, Σ^r is a closed submanifold of Σ^m and invariant under the action of S^3 , therefore there exists a tubular neighbourhood E of Σ^r which is invariant under the action of S^3 , so we may consider $\mu : E \rightarrow \Sigma^r$, the normal bundle to Σ^r in Σ^m and S^3 acts differentiably on E . See [[10],p.58]. Let S^{4k-1} be the boundary of a fibre of E , we can prove the following :

Lemma 2.2 $S^{4k-1} \subset \Sigma^m - \Sigma^r$ is a homotopy equivalence.

Proof. consider the exact cohomology sequence of the pair (Σ^m, Σ^r) :

$$\rightarrow H^{i-1}(\Sigma^r) \rightarrow H^i(\Sigma^m, \Sigma^r) \rightarrow H^i(\Sigma^m) \rightarrow H^i(\Sigma^r) \rightarrow H^{i+1}(\Sigma^m, \Sigma^r) \rightarrow H^{i+1}(\Sigma^m) \rightarrow$$

$$\text{At } i = m : \rightarrow 0 \rightarrow H^m(\Sigma^m, \Sigma^r) \rightarrow Z \rightarrow 0, \text{ then } H^m(\Sigma^m, \Sigma^r) \cong Z,$$

$$\text{At } i = r : 0 \rightarrow Z \rightarrow H^{r+1}(\Sigma^m, \Sigma^r) \rightarrow 0, \text{ hence } H^{r+1}(\Sigma^m, \Sigma^r) \cong Z,$$

$$\text{And for } i \neq m, r+1 : 0 \rightarrow 0 \rightarrow H^i(\Sigma^m, \Sigma^r) \rightarrow 0,$$

thus $H^i(\Sigma^m, \Sigma^r) \cong 0$ for $i \neq m, r+1$. Finally, we obtain

$$H^i(\Sigma^m, \Sigma^r) = \begin{cases} Z & \text{for } i = m, r+1 \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\Sigma^m - \Sigma^r$ is simply connected [[6],p.3] and using Lefschetz duality [[7],p.32], $H_i(\Sigma^m - \Sigma^r) \cong H^{m-i}(\Sigma^m, \Sigma^r)$, we finally deduce that

$$H_i(\Sigma^m - \Sigma^r) = \begin{cases} Z & \text{for } i = 0, 4k-1 \\ 0 & \text{otherwise} \end{cases}$$

hence, the inclusion map $i : S^{4k-1} \rightarrow \Sigma^m - \Sigma^r$ induces

$i_* : \pi_j(S^{4k-1}) \rightarrow \pi_j(\Sigma^m - \Sigma^r)$ an isomorphism $\forall j$, see[[13],p.283]. Therefore $S^{4k-1} \subset \Sigma^m - \Sigma^r$ is a homotopy equivalence.

Now, let $N = \Sigma^m - E_0$ where E_0 is the interior of an equivariant tubular neighbourhood of Σ^r with $\overline{E_0} \subset \text{int}(E)$. Then S^3 acts freely on N and $S^{4k-1} \subset N$. Notice that S^{4k-1} is a homotopy equivalence to N , it follows from the exact homotopy sequence of the fibre maps, using the diagram :

$$\begin{array}{ccccc} \rightarrow S^3 & \rightarrow & S^{4k-1} & \rightarrow & S^{4k-1}/S^3 \rightarrow \\ \downarrow & & \downarrow & & \downarrow \\ \rightarrow S^3 & \rightarrow & N & \rightarrow & N/S^3 \rightarrow \end{array}$$

that is $S^{4k-1}/S^3 \rightarrow N/S^3$ is a homotopy equivalence.

Set $N^1 = N/S^3$ and $S^{4k-1}/S^3 = QP(k-1)$. Notice that the region between ∂N^1 and $QP(k-1) \times S^r$ is an h-cobordism, so if $m \geq 6$, then by the h-cobordism theorem of Smale [14], N^1 is diffeomorphic to $QP(k-1) \times D^{r+1}$ and $N \rightarrow N^1$ is equivalent to $h \times I : S^{4k-1} \times D^{r+1} \rightarrow QP(k-1) \times D^{r+1}$, where $h : S^{4k-1} \rightarrow QP(k-1)$ is the hopf map.

Hence, we have proved the following theorem:

Theorem 2.2 Let (S^3, \sum^m, ϕ) be a semi-free S^3 action on \sum^m with fixed point set \sum^r , $m - r = 4k$, $k \geq 1$, $m \geq 6$. If N is the complement of an open tubular neighbourhood of \sum^r in \sum^m , then N is equivariantly diffeomorphic to $S^{4k-1} \times D^{r+1}$, with the standard action on S^{4k-1} and trivial action on D^{r+1} .

Now, we will describe how to construct smooth semi-free S^3 actions on a homotopy m -sphere \sum^m . Let \sum^r be a homotopy r -sphere and μ a (quaternionic) normal bundle over \sum^r given by $\mu : E(\mu) \rightarrow \sum^r$ where $E(\mu)$ is the total space of μ such that $E(\mu) \cong D^{4k} \times \sum^r$, i.e., $E(\mu)$ is the a trivial bundle and suppose that $h : S^{4k-1} \times \sum^r \rightarrow S^{4k-1} \times S^r$ is an equivariant diffeomorphism .

Theorem 2.3 There is a semi free action (S^3, \sum^m, ϕ) with a fixed point set \sum^r and $\sum^m = E(\mu) \cup_h (S^{4k-1} \times D^{r+1})$ where \cup_h means that we identify $S^{4k-1} \times \sum^r \subset E(\mu)$ with $S^{4k-1} \times S^r \subset S^{4k-1} \times D^{r+1}$ via the diffeomorphism h .

Proof. Consider the semi-free action on the total space of $\mu : E(\mu) \rightarrow \sum^r$ defined by the quaternionic structure and the free S^3 - action on $S^{4k-1} \times D^{r+1}$ defined by the free action on S^{4k-1} , i.e., the standard action and $h : S^{4k-1} \times \sum^r \rightarrow S^{4k-1} \times S^r$ is an equivariant diffeomorphism, then $M = E(\mu) \cup_h (S^{4k-1} \times D^{r+1})$ has a semi-free action of S^3 with fixed point set \sum^r and normal bundle μ , it is enough to show that M is a homotopy sphere.

It is clear that

$$\pi_1(\partial E(\mu)) \cong \pi_1(S^{4k-1} \times S^r) \cong \pi_1(S^{4k-1}) \oplus (S^r) \cong \pi_1(\sum^r) \text{ and } \pi_0(\partial E(\mu)) \cong \pi_0(\sum^r) \cong 0,$$

i.e., $E(\mu)$ and $S^{4k-1} \times D^{r+1}$ are simply connected and $E(\mu) \cap S^{4k-1} \times D^{r+1}$ is simply connected, hence by VanKampen's theorem [11], M is simply connected.

Now, we consider the Mayer -Vietoris sequence for M [2]:

$$\rightarrow H_{s+1}(M) \rightarrow H_s(\partial E(\mu)) \rightarrow H_s(E(\mu)) \oplus H_s(S^{4k-1} \times D^{r+1}) \rightarrow H_s(M) \rightarrow$$

By the Künneth formula [[9], p.98], since $H_s(\partial E(\mu))$, $H_s(E(\mu))$, $H_s(S^{4k-1} \times D^{r+1})$ are torsion free for, $0 < s < 4k + r - 1$, we obtain

$$H_s(\partial E(\mu)) = H_s(S^{4k-1} \times \sum^r) \cong \oplus_{i=0}^s H_i(S^{4k-1}) \otimes H_{s-i}(\sum^r).$$

If $i = 0$, then $H_0(S^{4k-1}) \otimes H_s(S^r) \cong Z \otimes H_s(\sum^r)$.

If $i = s$, then $H_s(S^{4k-1}) \otimes H_0(S^r) \cong H_s(S^{4k-1}) \otimes Z$ and for $i \neq 0$

$$H_i(S^{4k-1}) \otimes H_{n-i}(S^r) \cong 0 \text{ therefore } H_s(\partial E(\mu)) \cong Z \otimes H_s(S^r) \oplus H_s(S^{4k-1}) \otimes Z.$$

Again, we compute

$$H_s(S^{4k-1} \times D^{r+1}) = \oplus_{i=0}^s H_i(S^{4k-1}) \otimes H_{s-i}(D^{r+1}) \cong H_s(S^{4k-1}) \otimes Z.$$

similarly $H_s(E(\mu)) = H_s(D^{4k} \times \Sigma^r) \cong Z \otimes H_s(\Sigma^r)$.

therefore $H_s(\partial E(\mu)) \cong H_s(E(\mu)) \oplus H_s(S^{4k-1} \times D^{r+1})$,

hence $H_s(M) \cong H_{s+1}(M) \cong 0, \quad \forall 0 < s < 4k + r - 1$.

For, $s = 4k + r - 1$: $H_{4k+r-1}(E(\mu)) \cong 0$, $H_{4k+r-1}(S^{4k-1} \times D^{r+1}) \cong 0$ and $H_{4k+r-1}(\partial E(\mu)) \cong Z \otimes Z \cong Z$.

Substituting in the Mayer-Vietoris sequence for $M : 0 \rightarrow H_{4k+r}(M) \rightarrow Z \rightarrow 0$. Finally, we obtain

$$H_s(M) \cong \begin{cases} z & \text{for } s = 0, \quad 4k + r, \\ 0 & \text{otherwise} \end{cases}$$

hence M is a homology sphere.

But M is simply connected, closed without boundary, therefore M is a homotopy sphere.

3 Applying surgery to construct semi free S^3 – actions:

In this section, we used surgery techniques as Browder [16] to create a diffeomorphism of $QP(k-1) \times \Sigma^r$ with $QP(k-1) \times S^r$, then we apply Theorem 2.3

Theorem 3.1 Let Σ^{4n-1} be a homotopy sphere which bounds a parallelizable manifold, $n \geq 1$. Then for each even $k \geq 2$, there is a semi-free action of S^3 on a homotopy sphere $\Sigma^{4(n+k)-1}$ with Σ^{4n-1} as untwisted fixed point set.

Proof. Let $\Sigma^{4n-1} = \partial W^{4n}$, W is a parallelizable manifold. We may consider $W_0 = W - \text{int}(D^{4n})$ as a parallelizable cobordism between Σ^{4n-1} and S^{4n-1} thus we may define a normal map

$$f : (W_0, \Sigma^{4n-1} \cup S^{4n-1}) \rightarrow (S^{4n-1} \times I, S^{4n-1} \times \{0\} \cup S^{4n-1} \times \{1\})$$

with $f|_{S^{4n-1}} = \text{Identity}$. Since W is a parallelizable manifold [[8],p.514], we may assume that W_0 is $(2n-1)$ connected.

Multiplying by $QP(k-1)$, we get $I \times f$:

$$QP(k-1) \times (W_0, \Sigma^{4n-1} \cup S^{4n-1}) \rightarrow QP(k-1) \times (S^{4n-1} \times I, S^{4n-1} \times \{0\} \cup S^{4n-1} \times \{1\}),$$

with $I \times f|_{QP(k-1) \times S^{4n-1}} = \text{Identity}$.

The remainder of the proof is computing the obstruction σ for this map to be a cobordism and using this to determine if $QP(k-1) \times \Sigma^{4n-1}$ is diffeomorphic to $QP(k-1) \times S^{4n-1}$.

Claim: $Ker(I \times f)_* = H_*(QP(k-1)) \times Ker(f_*)$.

By the Künneth formula, since $H_*(QP(k-1))$ is torsion free then,

$$H_*(QP(k-1) \times (W_0, \Sigma^{4n-1} \cup S^{4n-1})) \cong H_*(QP(k-1)) \otimes H_*(W_0, \Sigma^{4n-1} \cup S^{4n-1})$$

and $(I \times f)_* = I \otimes f_*$

therefore, $Ker(I \times f)_* = H_*(QP(k-1)) \times Ker(f_*)$.

Now, consider the commuative diagram induced by f :

$$\begin{array}{ccccccc}
\longrightarrow H_i(\partial W_0) & \longrightarrow & H_i(W_0) & \longrightarrow & H_i(W_0, \partial W_0) & \longrightarrow & H_{i-1}(\partial W_0) \longrightarrow \\
\downarrow & & \downarrow & & f_* \downarrow & & \downarrow \\
\longrightarrow H_i(\partial S^{4n-1} \times I) & \longrightarrow & H_i(S^{4n-1} \times I) & \longrightarrow & H_i(S^{4n-1} \times I, \partial S^{4n-1} \times I) & \longrightarrow & H_{i-1}(\partial S^{4n-1} \times I) \longrightarrow
\end{array}$$

Notice that, $H_i(\partial S^{4n-1} \times I) \cong H_i(\partial W_0)$ and $H_i(W_0) \cong 0$ for $i \neq 0, 2n$. We get

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_{2n}(W_0) & \xrightarrow{\cong} & H_{2n}(W_0, \partial W_0) & \longrightarrow & \\
& & & & f_* \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Hence $Ker(f_*) \cong H_{2n}(W_0)$.

But $Ker(I \times f)_* = Ker(I \otimes f_*) = H_*(QP(k-1)) \otimes Ker(f_*) \cong H_*(QP(k-1)) \otimes H_{2n}(W_0)$, and $Ker(I \times f)_{*2n+2k-2} = H_{2k-2}(QP(k-1)) \otimes H_{2n}(W_0)$, since $2k-2 = 2(2s)-2 \neq 0 \pmod{4}$, k is even, then $H_{2k-2}(QP(k-1)) \cong 0$ and $Ker(I \times f)_{*2n+2k-2} \cong 0$.

Therefore, $\sigma(I \times f) = 0$ and there exists an h -cobordism between $QP(k-1) \times \sum^{4n-1}$ and $QP(k-1) \times S^{4n-1}$, but $k \geq 2$ and $n \geq 1$, then $(4k-4) + (4n-1) = 4(k+n)-5 \geq 7$. Hence, Smale's h -cobordism theorem can be applied and $QP(k-1) \times \sum^{4n-1}$ is diffeomorphic to $QP(k-1) \times S^{4n-1}$. Applying Theorem 2.3, it follows that there is a semi-free action of S^3 on some homotopy sphere \sum^m with \sum^{4n-1} as untwisted fixed point set, where $m = 4(n+k)-1$.

Open Problem :

Browder [16] showed how to construct semi-free S^1 actions, with \sum^r as untwisted fixed point set, i.e., its normal bundle is trivial. He stated that he did not know of any action with a twisted fixed point set. However Schultz [12] used complicated computations of homotopy groups, proved the following Theorem:

Let $k \geq 2$ be a positive integer. Then there exist infinitely many values of n for which S^{2n} has a semi-free S^1 action with $S^{2(n-k+1)}$ as twisted fixed point set. In this work, as in the work of Browder, we consider \sum^r as untwisted fixed point set and I raise the following question: Does there exist smooth semi free S^3 actions on homotopy spheres for which the fixed point set is twisted?

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